Diophantine equations, local-global principles and arithmetic statistics

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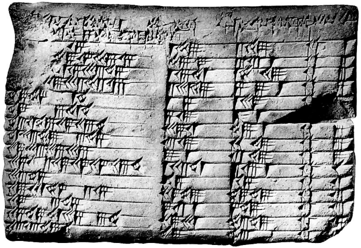
Diophantine equations



Mathematicians working on Diophantine equations study the integer solutions to polynomial equations with integer coefficients.

E.g. the Pythagorean equation $a^2 + b^2 = c^2$ has the integer solution $a = 3, b = 4, c = 5$.

Hilbert's 10th problem

Problem 10 (Hilbert, 1900)

Construct an algorithm which can decide whether any given Diophantine equation has a solution.

Hilbert's 10th problem

Theorem (Matiyasevich, Robinson, Davis, Putnam, 1970)

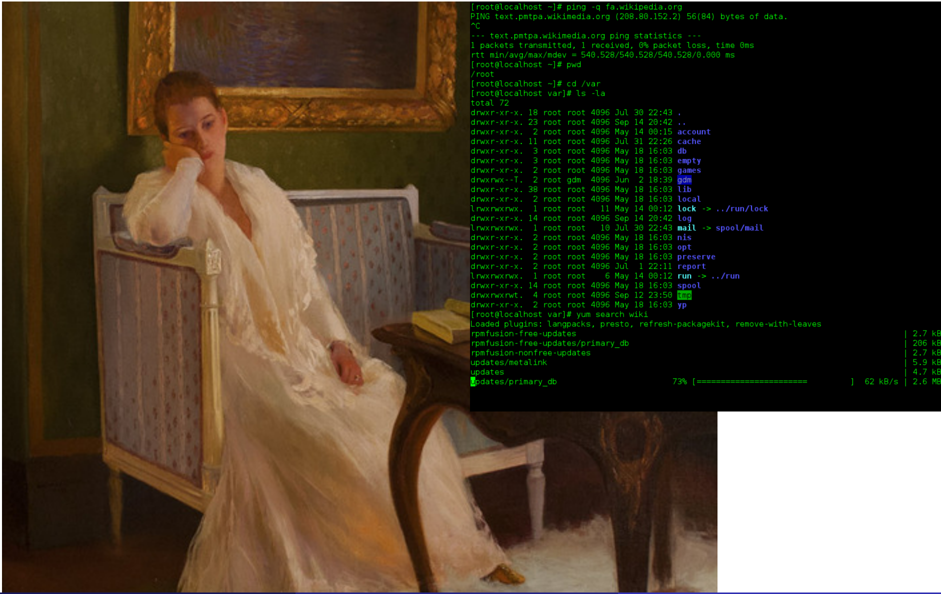
No such algorithm exists.

Question

Hilbert's 10th problem over $\mathbb{Q}$?

Wide open.

Searching for rational points



Proving no rational points exist

Let $X/\mathbb{Q}$ be an algebraic variety.

$$X(\mathbb{Q}) \subset X(\mathbb{R})$$

so

$$X(\mathbb{R}) = \emptyset \Longrightarrow X(\mathbb{Q}) = \emptyset.$$

$X(\mathbb{R})$ is easier to deal with than $X(\mathbb{Q})$ because $\mathbb{R}$ is complete.

The real world is not enough

But

$$X(\mathbb{R}) \neq \emptyset \not\Longrightarrow X(\mathbb{Q}) \neq \emptyset.$$

E.g. $x^2 = 2$ has real solutions but no rational solutions.

$\mathbb{R}$ is not the only completion of $\mathbb{Q}$.

The other completions are $\mathbb{Q}_p$, for p prime.

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$$

so

$$X(\mathbb{Q}_p) = \emptyset \Longrightarrow X(\mathbb{Q}) = \emptyset.$$

Idea: use all the completions of $\mathbb{Q}$ at once.

The Hasse principle

Let $X/\mathbb{Q}$ be a nice variety.

$$X(\mathbb{Q}) \subset X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p) =: X(\mathbb{A}_\mathbb{Q}) \quad \text{adelic points}$$

$$Q \mapsto (Q, Q, Q, Q, Q, \dots)$$

$$X(\mathbb{Q}) \neq \emptyset \Longrightarrow X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$$

Definition

If " $\Longleftarrow$ " holds, we say the **Hasse principle** holds.

What causes failures of the Hasse principle?

Brauer–Manin obstructions

Manin, 1970: Let $\mathrm{Br}\, X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. There's a pairing

$$X(\mathbb{A}_\mathbb{Q}) \times \mathrm{Br}\, X \quad \to \quad \mathbb{Q}/\mathbb{Z}$$

such that $\overline{X(\mathbb{Q})} \subset X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} :=$ adelic points orthogonal to $\mathrm{Br}\, X$.

- Suppose $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ but $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$. Then $X(\mathbb{Q}) = \emptyset$.
  Brauer–Manin obstruction to the Hasse principle

Weak approximation

Weak approximation holds if $X(\mathbb{Q})$ is dense in $X(\mathbb{A}_{\mathbb{Q}})$.

We have

$$\overline{X(\mathbb{Q})} \subset X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \subset X(\mathbb{A}_{\mathbb{Q}}).$$

- If $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq X(\mathbb{A}_{\mathbb{Q}})$ then $\overline{X(\mathbb{Q})} \neq X(\mathbb{A}_{\mathbb{Q}})$.
  Brauer–Manin obstruction to weak approximation

The Brauer–Manin pairing

The Brauer–Manin pairing is given by

$$\begin{array}{rcl} X(\mathbb{A}_{\mathbb{Q}}) \times \operatorname{Br} X & \to & \mathbb{Q}/\mathbb{Z} \\ ((Q_p)_p \ , \ \mathcal{A}) & \mapsto & \displaystyle\sum_{p \leq \infty} \mathcal{A}(Q_p) \end{array}$$

Let $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$ denote the set of adelic points orthogonal to $\mathcal{A} \in \operatorname{Br} X$.

Lemma

If $|\mathcal{A}| : X(\mathbb{Q}_v) \to \mathbb{Q}/\mathbb{Z}, Q_v \mapsto \mathcal{A}(Q_v)$, *is non-constant for some* v *then* $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} \neq X(\mathbb{A}_{\mathbb{Q}})$, *i.e.* $\mathcal{A}$ **obstructs weak approximation.**

Proof.

Let $(P_w)_w \in X(\mathbb{A}_{\mathbb{Q}})$. If $\sum_w \mathcal{A}(P_w) = 0$ then replace P_v with some Q_v such that $\mathcal{A}(Q_v) \neq \mathcal{A}(P_v)$. □

The Brauer group

The Brauer group has two parts:

- $\operatorname{Br}_1 X = \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X})$ "algebraic part" – easy to calculate
- $\operatorname{Br} X / \operatorname{Br}_1 X$ "transcendental part" – difficult to calculate, mostly unknown

Transcendental Brauer group calculations

- Conic bundles over $\mathbb{P}^2$ (Artin–Mumford, 1972)
- Diagonal quartic surfaces $ax^4 + by^4 + cz^4 + dw^4 = 0$ over $\mathbb{Q}$ (Ieronymou–Skorobogatov, 2014)
- Products $E \times E$ of CM elliptic curves (N., 2016)
- Non-diagonal quartic surfaces $ax^4 + bxy^3 + czw^3 + dz^4 = 0$ over $\mathbb{Q}$ (Alaa Tawfik–N., work in progress)

Example of a transcendental Brauer–Manin obstruction

Theorem (Alaa Tawfik–N., to appear)

Let $X/\mathbb{Q}$ be a Kummer surface with affine equation

$$w^2 = (x^3 + c)(t^3 + d).$$

$\operatorname{Br} X$ *contains a transcendental element of order* $5 \iff 80cd \in \mathbb{Q}(\zeta_3)^{\times 6}$.

Moreover, such an element always obstructs weak approximation.

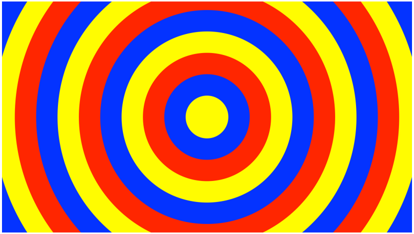
Uses work of Ieronymou–Skorobogatov where they obtain similar results for diagonal quartic surfaces.

How do the evaluation maps vary on p-adic discs?

Let $Q_p \in X(\mathbb{Q}_p)$.

- If $\mathcal{A}$ has order coprime to p then $\mathcal{A}(Q_p)$ only depends on Q_p mod p.
- If $\mathcal{A}$ has order p^n then $\mathcal{A}(Q_p)$ could depend on Q_p mod p^2 or mod p^3 etc.

Wild evaluation maps



Bright–N., 2020

For $\mathcal{A} \in \operatorname{Br} X$ of order p^n, we:

- calculate m such that $\mathcal{A}(Q_p)$ only depends on Q_p mod p^m
- show that $\mathcal{A}(Q_p)$ varies linearly on discs of points that are the same mod p^{m-1}
- if $p \mid m$, can get quadratic variation on larger discs

Which primes can be involved in the Brauer–Manin obstruction?

Let $\mathcal{A} \in \operatorname{Br} X$.

Question (Swinnerton-Dyer, 2010)

Suppose that $\operatorname{Pic} \bar{X}$ is torsion-free. Let p be a prime of good reduction for X (i.e. X mod p is smooth). Is $\mathcal{A}(Q_p)$ constant as Q_p varies in $X(\mathbb{Q}_p)$?

Equivalently, let $S = \{\text{primes of bad reduction}\} \cup \{\infty\}$. Does

$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = Z \times \prod_{p \notin S} X(\mathbb{Q}_p),$$

where $Z \subset \prod_{p \in S} X(\mathbb{Q}_p)$?

Does the Brauer–Manin obstruction involve only primes of bad reduction and infinite primes?

Which primes can be involved in the Brauer–Manin obstruction?

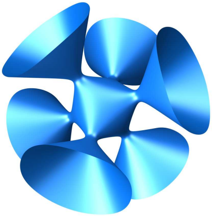
Theorem (Bright–N., 2020)

If $\mathrm{H}^0(X, \Omega^2_X) \neq 0$ *then every prime of good ordinary reduction is involved in a Brauer–Manin obstruction over some finite extension of the base field.*

Consequence:

The answer to Swinnerton-Dyer's question is no in general for K3 surfaces over number fields.

Image by Alessandra Sarti.



$1 + x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2 + 1)^2 = 0, \ a = -0.49$

Which primes can be involved in the Brauer–Manin obstruction?

Question

Suppose $\operatorname{Pic} \bar{X}$ is torsion-free. Is there a finite set S of primes that can be involved in the Brauer–Manin obstruction for X? Can we describe S?

Which primes can be involved in the Brauer–Manin obstruction?

Theorem (Bright–N., 2020)

Suppose $\operatorname{Pic} \bar{X}$ *is torsion-free. Then the finite set* S *consists of:*

- *primes of bad reduction;*
- *infinite primes;*
- *even primes;*
- *ramified primes;*
- *primes for which* $\mathrm{H}^0(X \bmod p, \Omega^1) \neq 0$ *(not needed if* X *is a K3 surface).*

E.g. for a K3 surface over $\mathbb{Q}$, the relevant primes are

$$\{\text{primes of bad reduction}\} \cup \{2, \infty\}.$$

How often does the Hasse principle fail?

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Family	**Proportion of failures**
$y^2 + z^2 = (at^2 + b)(ct^2 + d)$ (de la Bretèche–Browning, 2013)	0%
Hyperelliptic curves $z^2 = a_0 x^{2g+2} + a_1 x^{2g+1} y + \cdots + a_{2g+2} y^{2g+2}$ (Bhargava, 2013)	$> 0\%$ for $g = 1$, $> 50\%$ for $g \geq 2$, $> 99\%$ for $g \geq 10$
Plane cubics (Bhargava, 2014)	$> 0\%$ conjecturally $1 - 1/3$

Norm one tori

Let $L = \mathbb{Q}(\omega)$, degree d extension. The norm one torus for $L/\mathbb{Q}$ is the affine variety

$$T_{L/\mathbb{Q}} : N_{L/\mathbb{Q}}(x_0 + x_1\omega + \cdots + x_{d-1}\omega^{d-1}) = 1.$$

Its torsors are the affine varieties

$$T_{L/\mathbb{Q},\alpha} : \; N_{L/\mathbb{Q}}(x_0 + x_1\omega + \cdots + x_{d-1}\omega^{d-1}) = \alpha.$$

for $\alpha \in \mathbb{Q}^\times$.

Local and global points

$$T_{L/\mathbb{Q},\alpha}: \ N_{L/\mathbb{Q}}(x_0 + x_1\omega + \cdots + x_{d-1}\omega^{d-1}) = \alpha$$

$T_{L/\mathbb{Q},\alpha}(\mathbb{Q}) \neq \emptyset \iff \alpha$ is in the image of $N_{L/\mathbb{Q}} : L \to \mathbb{Q}$.

$T_{L/\mathbb{Q},\alpha}(\mathbb{Q}_p) \neq \emptyset \iff \alpha$ is in the image of $N_{L/\mathbb{Q}} : L \otimes \mathbb{Q}_p \to \mathbb{Q}_p$.

$T_{L/\mathbb{Q},\alpha}(\mathbb{R}) \neq \emptyset \iff \alpha$ is in the image of $N_{L/\mathbb{Q}} : L \otimes \mathbb{R} \to \mathbb{R}$.

Local and global points

Example

$L = \mathbb{Q}(i), \alpha = -2.$

$$N_{L/\mathbb{Q}}(x + yi) = (x + yi)(x - yi) = x^2 + y^2 = -2$$

$L \otimes \mathbb{R} = \mathbb{R}(i) = \mathbb{C}$. Image of $N_{L/\mathbb{Q}} : \mathbb{C}^\times \to \mathbb{R}^\times$ is $\mathbb{R}_{>0}$.

No real solution $\implies$ no rational (or "global") solution.

Local and global points

If $\alpha \neq 0$ is in the image of $N_{L/\mathbb{Q}} : L \to \mathbb{Q}$, say

"α is a global norm from $L/\mathbb{Q}$".

If $\alpha \neq 0$ is in the image of $N_{L/\mathbb{Q}} : L \otimes \mathbb{R} \to \mathbb{R}$ and in the image of $N_{L/\mathbb{Q}} : L \otimes \mathbb{Q}_p \to \mathbb{Q}_p$ for all p, say

"α is an everywhere local norm from $L/\mathbb{Q}$".

$$\{\text{global norms from } L/\mathbb{Q}\} \subset \{\text{everywhere local norms from } L/\mathbb{Q}\}.$$

The Hasse norm principle

$$Ш(T_{L/\mathbb{Q}}) = \frac{\{\text{everywhere local norms from } L/\mathbb{Q}\}}{\{\text{global norms from } L/\mathbb{Q}\}}$$

- If $Ш(T_{L/\mathbb{Q}}) = 1$ then the Hasse principle holds for all $T_{L/\mathbb{Q},\alpha}$ and we say the Hasse norm principle holds for $L/\mathbb{Q}$.
- If $Ш(T_{L/\mathbb{Q}}) \neq 1$ then there are rational numbers α which are everywhere locally norms from $L/\mathbb{Q}$ but not global norms. The Hasse principle fails for these $T_{L/\mathbb{Q},\alpha}$.

How often does the Hasse principle fail?

Family	Proportion of failures
$y^2 + z^2 = (at^2 + b)(ct^2 + d)$ (de la Bretèche–Browning, 2013)	0%
Hyperelliptic curves $z^2 = a_0 x^{2g+2} + a_1 x^{2g+1} y + \cdots + a_{2g+2} y^{2g+2}$ (Bhargava, 2013)	$> 0\%$ for $g = 1$, $> 50\%$ for $g \geq 2$, $> 99\%$ for $g \geq 10$
Plane cubics (Bhargava, 2014)	$> 0\%$ conjecturally $1 - 1/3$
Torsors for a norm one torus $T/\mathbb{Q}$ (Browning–N., 2016)	$1 - 1/\lvert Ш(T)\rvert$

Statistics of the Hasse norm principle

Let G be a finite abelian group.

A G-extension is a Galois extension with Galois group G.

Theorem (Frei–Loughran–N., 2018)

When ordered by conductor, 100% of G-extensions satisfy the Hasse norm principle.

We used this to give an asymptotic formula for the number of G-extensions from which a given element α is a norm.

S_4-quartics with prescribed norms

- Let $\alpha \in \mathbb{Q}^\times$.
- Write $N(B)$ for the number of S_4-quartic extensions $L/\mathbb{Q}$ with discriminant at most B.
- Write $N(B;\alpha)$ for the number of such extensions with $\alpha \in N_{L/\mathbb{Q}}(L^\times)$.

Theorem (Monnet, 2022)

$$0 < \lim_{B\to\infty} \frac{N(B;\alpha)}{N(B)} \leq 1,$$

with equality if and only if $\alpha \in \mathbb{Q}^{\times 4}$.

Statistics of the Hasse norm principle for non-abelian extensions

Theorem (N.–Varma, in preparation)

The Hasse norm principle holds for 100% of S_4-octics.

$$\mathcal{F}_{12} = \{\text{Degree } 12\ S_4\text{-fields fixed by a double transposition}\}.$$

The behaviour in this family is strikingly different from that of S_4-octics.

Theorem (N.–Varma, in preparation)

The Hasse norm principle fails for a positive proportion of fields in $\mathcal{F}_{12}$.

Thank you for your attention.